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Lifting three-dimensional wings in transonic flow

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The far field of a lifting three-dimensional wing in transonic flow is analysed. The boundary-value problem governing the flow far from the wing is derived by the method of matched asymptotic expansions. The main result is to show that corrections which are second order in the near field make a first-order contribution to the far field. The present study corrects and simplifies the work of Cheng & Hafez (1975) and Barnwell (1975).

1. Introduction

This paper is concerned with the transonic flow over thin lifting wings. In particular, the flow far from the wing is discussed. A number of authors have studied the transonic flow far from wings and bodies. One of the most important contributions to our understanding of these flows is the transonic area rule. This area rule (see, e.g. Oswatitsch 1952) states that the transonic flow far from a wing-body combination is the same as that produced by an equivalent body of revolution having the same axial distribution of cross-sectional area. This rule has been established for slender bodies by Oswatitsch (1952) and Cole & Messiter (1957). Spreiter & Stahara (1971) extended this to non-slender wings, i.e. wings having an aspect ratio of order one. Ashley & Landahl (1965) also extended the theory to include wings at an angle of attack comparable to their thickness. Generally, the area rule is deduced by deriving the boundary-value problem governing the flow far from the wing or body; this boundary-value problem is seen to be identical to that for a slender body of revolution, provided that the streamwise rate of change of cross-sectional area is the same for both.

Hayes (1954) has pointed out that the transonic area rule fails when the volume of the wing is sufficiently small. Cheng & Hafez (1972, 1973, 1975) and Barnwell (1973, 1975) have studied the effect of lift on the transonic area rule. The study presented here treats the case of a lifting wing with no thickness. With this simple case it is easy to illustrate the basic theory and the main effect of lift on the far field. The boundary-value problem governing the flow far from the wing is obtained through a straightforward application of the method of matched asymptotic expansions.

Our main interest here is in the basic theory of lifting wings in transonic flow; for a more complete discussion of the flow from a physical point of view, and of extensions to the basic theory, we refer the reader to the references cited above.

In §6 we review and discuss the previous investigations of Cheng & Hafez (1975) and Barnwell (1975). In most respects our work agrees with the above authors. However, there are important differences in our expressions for the boundary condition

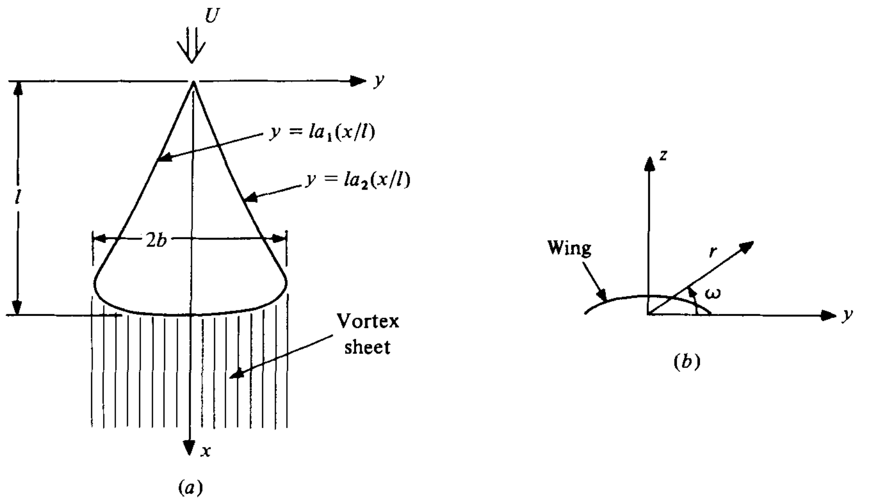

FIGURE 1. Sketch of wing and co-ordinate system. (a) Plan view; (b) rear view, $r \equiv (z^2+y^2)^{\frac{1}{2}}$.

for the outer flow; the disagreement with the results of Cheng & Hafez is due to fundamental differences in the matching.

2. Mathematical formulation

The co-ordinate system used is sketched in figure 1; the origin is taken at the nose of the wing, the x axis is aligned with the undisturbed uniform flow, and the z axis is taken as approximately perpendicular to the wing surface. A typical wing has been sketched; it has a length l and a span of $2b$. The equations defining the wing are

$$z = \mathscr{Z}(x,y;l;b;\alpha) = \alpha l Z\left(\frac{x}{l},\frac{y}{l};\frac{b}{l}\right)$$

for $a_1(x/l) \leqslant y/l \leqslant a_2(x/l)$; the functions a_1 and a_2 give the leading edges of the wing as well as the outer edges of the trailing vortex sheet. The function Z is taken to be some sufficiently smooth function of x and y. Because it defines a single surface in space, the wing has no thickness. The aspect ratio is taken to be of order one; i.e. b/l will be assumed to be of order one. To eliminate unnecessary writing, the independent variables x, y and z will be scaled by l; otherwise the quantity $b/l = O(1)$ will appear throughout the calculations. For the sake of simplicity Cheng & Hafez (1975) assume that Z is such that there are no singularities at the leading edges. In fact, they assume that the velocity perturbations are zero at the leading edges and the outer edges of the trailing vortex sheet; here we assume this as well. By making this assumption we avoid the difficulties associated with leading edge singularities and separation addressed by Barnwell (1975). The small parameters of the problem are α, which gives a measure of the angle of attack of the wing, and M_0^2-1, which indicates that the flow is transonic; here $M_0 \equiv U/a_0$, where U is the speed of the undisturbed uniform flow and a_0 is the ambient sound speed.

The flow is assumed to be irrotational; a velocity potential ϕ^* therefore exists and the inviscid equations of motion of a perfect gas may be written

$$\nabla\phi^* . \nabla\left(\frac{\nabla\phi^* . \nabla\phi^*}{2}\right) = a^2\nabla^2\phi^*,$$

where

$$a^2 \equiv a_0^2 + \frac{\gamma-1}{2}\left[U^2 - \frac{\nabla\phi^* . \nabla\phi^*}{2}\right],$$

a is the local speed of sound in the gas and the γ is the ratio of specific heats. The last equation is just the Bernoulli equation for steady isentropic flow. The velocity potential ϕ^* contains a part due to the uniform stream and a part due to the perturbation of the wing. It will be convenient to work with the equation for the perturbation potential, ϕ, defined by

$$\phi^* \equiv Ux + \phi.$$

In terms of ϕ, the equation of motion is

$$\left.\begin{aligned} U^2\phi_{xx} + U\frac{\partial}{\partial x}|\nabla\phi|^2 + \nabla\phi . \nabla\left(\frac{|\nabla\phi|^2}{2}\right) &= a^2\nabla^2\phi, \\ a^2 = a_0^2 - (\gamma-1)\left[U\phi_x + \frac{|\nabla\phi|^2}{2}\right]&, \end{aligned}\right\} \tag{1}$$

exactly. The boundary condition on the wing is

$$\phi_z = (U+\phi_x)\,\mathscr{Z}_x + \phi_y\,\mathscr{Z}_y \tag{2}$$

on $z = \mathscr{Z}(x,y;l;b;\alpha)$. As $x^2+y^2+z^2 \to \infty$, it is further required that $|\nabla\phi| \to 0$.

In the following sections, solutions to (1) and (2) are sought which are valid for small α and $M_0^2 - 1$. In § 3 the solution valid near the wing is derived; this will be called the inner solution. Because the inner solution neglects certain nonlinear terms in (1), it fails to give a valid description of the flow at large distances from the wing. An approximation to (1) which is valid at large distances from the wing is derived in § 4. There it is shown that, to lowest order, the flow is governed by the small disturbance transonic equation; the region in which this is valid is called the outer region. The boundary condition satisfied by the first term of the outer expansion is obtained by a matching with the inner solution; this is done in § 5. There it is seen that every term in the inner solution contributes to the boundary condition for the outer problem; the resultant boundary condition for the outer problem will therefore be an infinite sum of terms. The outer expansion will be written

$$\phi = Ulf_0\,\Phi_0(\tilde{x},\hat{r},\omega) + o(f_0),$$

where $\tilde{x}$ and $\hat{r}$ are just scaled values of x and r (see figure 1*b*); the function Φ_0 will be shown to satisfy

$$\left.\begin{aligned} \Phi_{0\hat{r}\hat{r}} + \frac{1}{\hat{r}}\Phi_{0\hat{r}} + \frac{1}{\hat{r}^2}\Phi_{0\omega\omega} &= \frac{M_0^2-1}{f_0}\Phi_{0\tilde{x}\tilde{x}} + (\gamma+1)\Phi_{0\tilde{x}}\Phi_{0\tilde{x}\tilde{x}}, \\ \Phi_0(\tilde{x},\hat{r},\omega) &\sim \frac{a(\tilde{x})}{\hat{r}} + b(\tilde{x})\,[\ln^2\hat{r} + \cos^2\omega] + c(\tilde{x})\ln\hat{r} + d(\tilde{x}) + \ldots, \end{aligned}\right\} \tag{3}$$

as $\hat{r} \to 0$, and $\hat{r}^{-1}\Phi_{0\omega}$, $\Phi_{0\hat{r}}$, $\Phi_{0\tilde{x}} \to 0$, as $\hat{r} \to \infty$. Here the dots indicate terms which are of order $\hat{r}\ln^3\hat{r}$ and higher in $\hat{r}$. An analogous result has been obtained by Cole & Messiter (1957) for the case of a slender axisymmetric body in transonic flow, see, e.g. their equation 5.9. Note also that the terms shown in (3) are singular as $\hat{r}\to 0$, whereas the unwritten terms vanish in this limit. In §5, it is further assumed that Φ_0 may be uniquely determined by a specification of the singularities at the axis; we may therefore truncate the infinite series and write the boundary condition as

$$\Phi_0 \sim \frac{a}{\hat{r}} + b[\ln^2\hat{r} + \cos^2\omega] + c\ln\hat{r} + d$$

as $\hat{r}\to 0$.

3. Inner solution

In the inner region the velocity potential ϕ and the independent variables x, y and z may be scaled as follows:

$$\phi = Ul\varphi, \quad x = l\tilde{x}, \quad y = l\tilde{y}, \quad z = l\tilde{z},$$

provided $b/l = O(1)$; where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are of order one in the inner region. Equations (1) and (2) may now be written as

$$\left.\begin{aligned} M_0^2\left[\varphi_{\tilde{x}\tilde{x}} + \frac{\partial}{\partial\tilde{x}}|\nabla\varphi|^2 + \nabla\varphi\,.\,\nabla\left(\frac{|\nabla\varphi|^2}{2}\right)\right] &= \frac{a^2}{a_0^2}\nabla^2\varphi,\\ \frac{a^2}{a_0^2} &= 1-(\gamma-1)M_0^2\left(\varphi_{\tilde{x}} + \frac{|\nabla\varphi|^2}{2}\right)\end{aligned}\right\} \tag{4}$$

with

$$\varphi_{\tilde{z}} = \alpha[(1+\varphi_{\tilde{x}})Z_{\tilde{x}} + \varphi_{\tilde{y}}Z_{\tilde{y}}] \quad \text{on} \quad \tilde{z} = \alpha Z(\tilde{x},\tilde{y}), \tag{5}$$

where all derivatives are now with respect to $\tilde{x}$, $\tilde{y}$, $\tilde{z}$.

Equations (4) and (5) will now be perturbed for small α and M_0^2-1; the inner expansion is written

$$\varphi^i = g_0\varphi_0 + g_1\varphi_1 + O(g_2), \tag{6}$$

where the g_i's are the as yet undetermined inner gauge functions. When (6) is substituted into (4) and (5) and when the coefficients of like powers of α and M_0^2-1 are equated, there results a set of boundary-value problems, each of which is of the form

$$\left.\begin{aligned} \nabla_2^2\psi &= \mathscr{F}(\tilde{y},\tilde{z})\\ \psi_{\tilde{z}}(\tilde{y},0\pm) &= f^{\pm}(\tilde{y}), \quad a_1 \leqslant \tilde{y} \leqslant a_2,\end{aligned}\right\} \tag{7}$$

where $\nabla_2^2 \equiv \partial^2/\partial\tilde{z}^2 + \partial^2/\partial\tilde{y}^2$; here $\tilde{x}$ only appears as a parameter and its dependence has not been explicitly shown. At this point it is useful to review the method of solution of (7); this will not only give a simple formula for the solution, but will also clarify certain of its features.

Solutions to (7) are not unique; the operator is elliptic, but boundary values are only specified on a slit $\tilde{z} = 0\pm$, $a_1 \leqslant \tilde{y} \leqslant a_2$. It is easily seen that any two solutions of (7) differ by, at most, a harmonic function. In this paper the arbitrary harmonic function is determined by matching to the outer solution. We first decompose the solution to (7) as follows:

$$\psi = \psi_p + \psi_H, \tag{8}$$

where ψ_p satisfies

$$\nabla_2^2 \psi_p = \mathscr{F}(\tilde{y}, \tilde{z})$$

and ψ_H satisfies

$$\nabla_2^2 \psi_H = 0,$$

$$\psi_{H\tilde{z}} = f^{\pm} - \psi_{p\tilde{z}} \quad \text{on} \quad \tilde{z} = 0\pm, \quad a_1 \leqslant \tilde{y} \leqslant a_2.$$

In terms of the complex variables $\zeta \equiv \tilde{y} + i\tilde{z}$ and $\bar{\zeta} \equiv \tilde{y} - i\tilde{z}$, the above equation for ψ_p may be written

$$\psi_{p\zeta\bar{\zeta}} = \tfrac{1}{4}\mathscr{F}(\zeta, \bar{\zeta});$$

thus ψ_p may be obtained by integrating with respect to ζ and $\bar{\zeta}$:

$$\psi_p = \frac{1}{4}\iint \mathscr{F}(\zeta, \bar{\zeta})\, d\zeta\, d\bar{\zeta} + \mathscr{H}_p,$$

where $\mathscr{H}_p$ is any harmonic function and is composed of the two arbitrary functions of integration. Because the function $\mathscr{F}$ is given, the indefinite integral $\iint \mathscr{F} d\zeta\, d\bar{\zeta}$ can be calculated explicitly. At this stage, it is convenient, but not necessary, to choose $\mathscr{H}_p$; the choice of $\mathscr{H}_p$ will only affect ψ_H and not the final result for ψ. This will be chosen such that ψ_p is some simple known function, say Ψ, e.g. when $\mathscr{F} \equiv 0$, $\mathscr{H}_p$ will be taken to be zero as well. We now discuss the harmonic part of the solution ψ_H. Because no conditions at infinity are given, ψ_H will be arbitrary; we may always rewrite ψ_H as

$$\psi_H = \psi_H' + \mathscr{H}_H,$$

where ψ_H' is defined by

$$\nabla_2^2 \psi_H' = 0,$$

$$\psi_{H\tilde{z}}' = f^{\pm} - \Psi_{\tilde{z}} \quad \text{on} \quad \tilde{z} = 0\pm, \quad a_1 \leqslant \tilde{y} \leqslant a_2,$$

and

$$\psi_H' \sim b_0 \ln \tilde{r} + b_1 + \dots \quad \text{as} \quad \tilde{r} \to \infty.$$

The function $\mathscr{H}_H$ may be any function satisfying

$$\nabla_2^2 \mathscr{H}_H = 0,$$

$$\mathscr{H}_{H\tilde{z}} = 0 \quad \text{on} \quad \tilde{z} = 0\pm, \quad a_1 \leqslant \tilde{y} \leqslant a_2.$$

The solution for ψ_H' is well known (see, e.g. Ashley & Landahl 1965):

$$\psi_H' = \frac{1}{2\pi}\int_{a_1}^{a_2} [\psi_{H\tilde{z}}'] \ln r_1\, dy + \frac{1}{2\pi}\int_{a_1}^{a_2} [\psi_H'] \frac{\tilde{z}}{r_1^2}\, dy_1 + K,$$

where $r_1 \equiv \{(y_1 - \tilde{y})^2 + \tilde{z}^2\}^{\frac{1}{2}}$ and the square brackets indicate the jump in the quantity across the slit. Thus, ψ_H is given by

$$\psi_H = \frac{1}{2\pi}\int_{a_1}^{a_2} [f - \Psi_{\tilde{z}}] \ln r_1\, dy_{11} + \frac{1}{2\pi}\int_{a_1}^{a_2} [\psi_H] \frac{\tilde{z}}{r_1^2}\, dy_1 + K + \mathscr{H}_H - \frac{1}{2\pi}\int_{a_1}^{a_2} [\mathscr{H}_H] \frac{\tilde{z}}{r_1^2}\, dy_1.$$

When this is substituted in (8) and the fact that $[\psi_H] = [\psi - \Psi]$ is used, we have

$$\psi = \Psi + \frac{1}{2\pi}\int_{a_1}^{a_2} [f - \Psi_{\tilde{z}}] \ln r_1\, dy_1 + \frac{1}{2\pi}\int_{a_1}^{a_2} [\psi - \Psi] \frac{\tilde{z}}{r_1^2}\, dy_1 + K + \mathscr{H}_H - \frac{1}{2\pi}\int_{a_1}^{a_2} [\mathscr{H}_H] \frac{\tilde{z}}{r_1^2}\, dy_1. \tag{9}$$

Thus, (9) gives the desired solution to (7) in terms of the known functions $f^{\pm}(\tilde{y})$ and $\mathscr{F}(\tilde{y},\tilde{z})$ and the unknown harmonic function $\mathscr{H}_H$. It is clear from the above results that when $\mathscr{F}$ is not identically zero, the particular solution will induce a source or doublet distribution on the slit; this is due to the fact that the second and third terms in (9) contain $[\Psi_{\tilde{z}}]$ and $[\Psi]$ in their integrands. In the theory presented here the constant K and the harmonic function $\mathscr{H}_H(\tilde{y},\tilde{z})$ may also be functions of $\tilde{x}$ and α.

We now return to the equations (4) and (5); when (6) is substituted in (4) and (5), we find that

$$\nabla_2^2 \varphi_0 = 0, \quad \varphi_{0\tilde{z}} = Z_{\tilde{x}}(\tilde{x},\tilde{y}) \quad \text{on} \quad \tilde{z} = 0\pm, \quad a_2 \leqslant \tilde{y} \leqslant a_2,$$

provided we choose $g_0 = \alpha$. The solution to this is given by (9), with $\Psi \equiv 0$:

$$\varphi_0 = \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_{0\tilde{z}}] \ln r_1 \, dy_1 + \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_0 - \mathscr{H}_0] \frac{\tilde{z}}{r_1^2} dy_1 + K_0(x) + \mathscr{H}_0(x,y,z).$$

Because $\varphi_{0\tilde{z}}$ is continuous across the wing the first integral may be dropped. Furthermore, the above boundary-value problem is satisfied by functions φ_0 which are antisymmetric in $\tilde{z}$, i.e.

$$\varphi_0(\tilde{z}) = -\varphi_0(-\tilde{z}).$$

Here we will assume that both φ_0 and $\mathscr{H}_0$ have this symmetry; hence, $K_0(\tilde{x}) \equiv 0$ and

$$\varphi_0(\tilde{x},\tilde{y},\tilde{z}) = \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_0 - \mathscr{H}_0] \frac{\tilde{z}}{r_1^2} dy_1 + \mathscr{H}_0.$$

In order to obtain higher order terms in the inner expansion, it will be convenient to anticipate some of the results of matching to the outer solution. The inner expansion of the outer solution is essentially the inner boundary condition for the outer problem; this must be matched to the large $\tilde{r}$ expansion of φ_0:

$$\varphi_0 \sim \frac{1}{2\pi}\frac{\sin\omega}{\tilde{r}}\int_{a_1}^{a_2} [\varphi_0 - \mathscr{H}_0]\, dy_1 + \mathscr{H}_0 + O(\tilde{r}^{-2}),$$

where $\mathscr{H}_0$ has not yet been expanded and $\sin\omega = \tilde{z}/\tilde{r}$, $\cos\omega = \tilde{y}/\tilde{r}$. We require that this boundary condition contain at least the doublet

$$\frac{1}{2\pi}\frac{\sin\omega}{\tilde{r}}\int_{a_1}^{a_2} [\varphi_0]\, dy_1;$$

this will only be possible if, at large $\tilde{r}$, $\mathscr{H}_0 = O(\tilde{r}^{-1})$, at most. Thus, the matching gives the large $\tilde{r}$ behaviour of $\mathscr{H}_0$; $\mathscr{H}_0$ therefore satisfies

$$\nabla^2 \mathscr{H}_0 = 0, \quad \mathscr{H}_{0\tilde{z}} = 0 \quad \text{on} \quad \tilde{z} = 0, \quad a_1 \leqslant \tilde{y} \leqslant a_2,$$

$$\mathscr{H}_0 = O(\tilde{r}^{-1}) \quad \text{as} \quad \tilde{r} \to \infty,$$

which implies that

$$\mathscr{H}_0 = \frac{1}{2\pi}\int_{a_1}^{a_2} [\mathscr{H}_0] \frac{\tilde{z}}{r_1^2} dy_1.$$

Thus,

$$\varphi_0(\tilde{x},\tilde{y},\tilde{z}) = \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_0] \frac{\tilde{z}}{r_1^2} dy_1. \tag{10}$$

This gives φ_0 in terms of its jump across the wing and trailing vortex sheet. This jump is not known *a priori*; (10) leads to an integral equation for $[\varphi_0]$. Because the primary concern here is the flow far from the wing, it will be assumed that this integral equation has been solved and that $[\varphi_0]$ is known everywhere on the wing and in the trailing vortex sheet.

For the purposes of matching, the large $\tilde{r}$ behaviour of φ_0 is of interest; this is

$$\varphi_0 \sim \frac{F(\tilde{x})}{2\pi}\frac{\sin\omega}{\tilde{r}} + O\left(\frac{1}{\tilde{r}^2}\right), \tag{11}$$

where

$$F(\tilde{x}) \equiv \int_{a_1}^{a_2} [\varphi_0]\,dy_1.$$

This is immediately recognized as the potential due to a line doublet.

We now discuss the solution for φ_1. Substitution of the inner expansion (6) in (4) yields

$$\begin{aligned}\varphi_{\tilde{y}\tilde{y}} + \varphi_{\tilde{z}\tilde{z}} = {} & \alpha^2 \frac{\partial}{\partial\tilde{x}}\left[\frac{\gamma+1}{2}\varphi_{0\tilde{x}}^2 + \varphi_{0\tilde{y}}^2 + \varphi_{0\tilde{z}}^2\right] + \alpha g_1\left\{2\frac{\partial}{\partial\tilde{x}}\left[\frac{\gamma+1}{2}\varphi_{0\tilde{x}}\varphi_{1\tilde{x}}\right.\right. \\ & \left.\left. + \varphi_{0\tilde{y}}\varphi_{1\tilde{y}} + \varphi_{0\tilde{z}}\varphi_{1\tilde{z}}\right] + (\gamma-1)\varphi_{0\tilde{x}}\nabla_2^2\varphi_1 + (M_0^2-1)\alpha\varphi_{0\tilde{x}\tilde{x}}\right\} \\ & + \alpha^3\left\{\frac{\gamma+1}{2}\varphi_{0\tilde{x}}^2\varphi_{0\tilde{x}\tilde{x}} + \frac{\gamma-1}{2}\varphi_{0\tilde{x}\tilde{x}}(\varphi_{0\tilde{y}}^2 + \varphi_{0\tilde{z}}^2) + \varphi_{0\tilde{y}}^2\varphi_{0\tilde{y}\tilde{y}} + \varphi_{0\tilde{z}}^2\varphi_{0\tilde{z}\tilde{z}}\right. \\ & \left. + 2\varphi_{0\tilde{x}}\varphi_{0\tilde{y}}\varphi_{0\tilde{x}\tilde{y}} + 2\varphi_{0\tilde{x}}\varphi_{0\tilde{z}}\varphi_{0\tilde{x}\tilde{z}} + 2\varphi_{0\tilde{y}}\varphi_{0\tilde{z}}\varphi_{0\tilde{y}\tilde{z}}\right\} \\ & + O[(M_0^2-1)\alpha^2,\ (M_0^2-1)g_1] + o(\alpha^3, \alpha g_1),\end{aligned} \tag{12}$$

where use has been made of the fact that $\nabla_2^2\varphi_0 = 0$ and that $g_0 = \alpha$. In a similar manner the boundary condition (5) may be expanded to yield

$$\begin{aligned}\varphi_{\tilde{z}} = {} & \alpha Z_{\tilde{x}} - \alpha^2\{Z_{\tilde{x}}\varphi_{0\tilde{x}} + (Z\varphi_{0\tilde{y}})_{\tilde{y}}\} + \alpha g_1\{Z_{\tilde{x}}\varphi_{1\tilde{x}} + Z_{\tilde{y}}\varphi_{1\tilde{y}} - Z\varphi_{1\tilde{z}\tilde{z}}\} \\ & + \alpha^3\left\{ZZ_{\tilde{x}}\varphi_{0\tilde{x}\tilde{z}} + \left(\frac{Z^2}{2}\varphi_{0\tilde{y}\tilde{z}}\right)_{\tilde{y}}\right\} + o(\alpha^3, \alpha g_1),\end{aligned} \tag{13}$$

at $\tilde{z} = 0$. Here the usual Taylor series' expansions have been used to transfer the boundary condition from the wing surface to the $\tilde{z} = 0$ plane. In order to save space, the left-hand sides of both equations (12) and (13) have been left in terms of the exact potential φ; these of course, must be expanded in (6) when the actual calculations are performed.

At this stage, it is necessary to discuss the size of the $(M_0^2-1)\alpha\varphi_{0\tilde{x}\tilde{x}}$ term appearing in (12). In many theories of transonic aerodynamics, the matching of the near- and far-field solutions establishes a relationship between M_0^2-1 and the thickness or angle of attack of the wing or body. If, in the present case, we were to make no assumption about the size of M_0^2-1, the matching would show that

$$M_0^2 - 1 = O(\delta^2),$$

where δ is the ratio of the inner and outer length scales and is related to the angle of attack, α, through the equation

$$\alpha = \frac{\delta}{|\ln\delta|^{\frac{1}{2}}}.$$

In the following, we shall anticipate this result and use it wherever it is convenient; the reason for doing this is to keep the discussion of the inner expansion as concrete as possible.

We may now identify g_1 and the boundary-value problem for φ_1; this is

$$\nabla_2^2 \varphi_1 = \frac{\partial}{\partial \tilde{x}}\left[\frac{\gamma+1}{2}\varphi_{2\tilde{x}}^2 + \varphi_{0\tilde{y}}^2 + \varphi_{0\tilde{z}}^2\right] \tag{14}$$

with

$$\varphi_{1\tilde{z}}(\tilde{y}, 0\pm) = Z_{\tilde{x}}\,\varphi_{0\tilde{x}} + (Z\varphi_{0\tilde{y}})_{\tilde{y}}$$

for $a_1(\tilde{x}) \leqslant \tilde{y} \leqslant a_2(\tilde{x})$ provided we take $g_1 = \alpha^2$. In appendix A, the result (9) has been used to solve equations (14) for φ_1. Under the assumptions stated there φ_1 may be written

$$\varphi_1 = \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_{0\tilde{y}}]\, Z\frac{\tilde{y}-y_1}{r_1^2}\,dy_1 + \tfrac{1}{2}(\varphi_0^2)_{\tilde{x}} - \frac{1}{2\pi}\int_{a_1}^{a_2}[\varphi_0]\,Z_{\tilde{x}\tilde{x}}\ln r_1\,dy_1$$
$$+\frac{\gamma+1}{2}\left\{\Psi''' - \frac{1}{2\pi}\int_{a_1}^{a_2}[\Psi'''_{\tilde{z}}]\ln r_1\,dy_1\right\} + \mathscr{K}(\tilde{x}),$$

and the large $\tilde{r}$ expansion of φ_1 is

$$\varphi_1 \sim \frac{S'(\tilde{x})}{2\pi}\ln\tilde{r} + \mathscr{K}(\tilde{x}) + \frac{\gamma+1}{32\pi^2}(F'^2)'(\ln^2\tilde{r} + \cos^2\omega) + O\left(\frac{\ln\tilde{r}}{\tilde{r}}, \frac{1}{\tilde{r}}\right),$$

where Ψ''' and $\psi'''_{\tilde{z}}(\tilde{y}, 0\pm; \tilde{x})$ are given by equations (A 7) and (A 5) respectively, and

$$S'(\tilde{x}) \equiv \frac{dS(\tilde{x})}{d\tilde{x}} \equiv G(\tilde{x}) + \frac{\gamma+1}{2}I(\tilde{x}),$$

$$G(\tilde{x}) \equiv -\int_{a_1}^{a_2}[\varphi_0]\,Z_{\tilde{x}\tilde{x}}\,dy_1, \quad I(\tilde{x}) \equiv -\int_{a_1}^{a_2}[\Psi'''_{\tilde{z}}]\,dy_1,$$

and $\mathscr{K}(\tilde{x})$ is an arbitrary function of $\tilde{x}$; it may also have a weak, e.g. logarithmic, dependence on α.

Inspection of (A 10) shows that φ_1 contains a source-like term as well as one which depends nonlinearly on the lift $F'(\tilde{x})$. Because $[\varphi_0] \neq 0$ on the trailing vortex sheet, $G(\tilde{x}) \neq 0$ there; hence, the source has an afterbody associated with it. The results obtained here are equivalent to those obtained previously by Cheng & Hafez (1975).

It is clear from equations (11) and (A 10) that the inner expansion (6) breaks down at large values of $\tilde{r}$; this is because nonlinear terms in (4) play an important role far from the wing. In the next section the nonlinear equation governing the flow far from the wing is derived.

4. Outer region

Far from the wing the $\tilde{y}$ and $\tilde{z}$ co-ordinates must be stretched relative to the $\tilde{x}$ co-ordinate; thus, we define the outer variables $\hat{y}$ and $\hat{z}$ by

$$\hat{y} \equiv \delta(\alpha)\,\tilde{y}, \quad \hat{z} \equiv \delta(\alpha)\,\tilde{z},$$

where $\delta = o(1)$ as $\alpha \to 0$. The outer expansion is written

$$\varphi^0 = f_0(\alpha)\,\Phi_0(\tilde{x}, \hat{y}, \hat{z}) + o(f_0).$$

For the sake of simplicity we shall assume that Φ_0 is also a weak function of α or $\delta(\alpha)$, e.g. logarithmic in δ. The results we obtain will be in accord with those found by Cheng & Hafez (1975). The advantage of the procedure used here is that we need only to discuss a one term outer expansion. Barnwell (1975) has approached the outer expansion from a different point of view; he primarily discusses three terms of an outer expansion having gauge functions which may be written

$$f_0, \frac{f_0}{\sqrt{|\ln \delta|}}, \frac{f_0}{[\ln \delta|}, \ldots,$$

where the coefficients of these gauge functions are now independent of α. The dots indicate the higher-order gauge functions; an inspection of higher-order terms suggests that this is an infinite sequence of terms of the general form $f_0/|\ln \delta|^{\frac{1}{2}n}$, where n takes on integral values. The relationship between the two approaches is clear; the outer expansion of Barnwell results from expanding our $\Phi_0(\tilde{x}, \tilde{r}, \omega; \ln \delta)$ for small δ. One can show that the results obtained by either approach are equivalent to the appropriate order.

When this scaling and outer expansion are substituted into the exact equation of motion (4), this equation becomes

$$\delta^2 f_0(\Phi_{0\hat{y}\hat{y}} + \Phi_{0\hat{z}\hat{z}}) = (M_0^2 - 1) f_0 \Phi_{0\tilde{x}\tilde{x}} + f_0^2 \frac{\partial}{\partial \tilde{x}}\left(\frac{\gamma+1}{2} \Phi_{0\tilde{x}}^2\right) + o(\delta^2 f_0, (M_0^2 - 1) f_0, f_0^2).$$

We now require that the four terms which are shown explicitly balance in the outer region; thus

$$\delta = \sqrt{f_0} \quad \text{and} \quad M_0^2 - 1 = O(f_0) = O(\delta^2),$$

and the equation satisfied by Φ_0 is

$$\Phi_{0\hat{y}\hat{y}} + \Phi_{0\hat{z}\hat{z}} = \frac{M_0^2 - 1}{\delta^2} \Phi_{0\tilde{x}\tilde{x}} + \frac{\partial}{\partial \tilde{x}}\left(\frac{\gamma+1}{2} \Phi_{0\tilde{x}}^2\right). \tag{15}$$

The outer equation is immediately recognized as the three-dimensional, small disturbance, transonic equation. The boundary condition for this equation must come from a matching with the inner solution; this we carry out in the next section. In addition to providing the boundary condition for the outer problem, the matching determines the scale factor δ explicitly in terms of α.

5. The matching

In this section the inner and outer expansions are matched. For the sake of simplicity, Van Dyke's (1964) matching principle is used. The more sophisticated technique of intermediate expansions gives results identical to the ones presented here.

The two term inner expansion reads

$$\varphi^i = \alpha\varphi_0 + \alpha^2\varphi_1 + o(\alpha^2),$$

where φ_0 and φ_1 have been given explicitly by equations (10) and (A 9), respectively. The one term outer expansion is given by

$$\varphi^0 = f_0 \Phi_0(\tilde{x}, \hat{y}, \hat{z}) + o(f_0),$$

where $\hat{y} = \delta(\alpha)\tilde{y}$, $\hat{z} = \delta(\alpha)\tilde{z}$ and $\delta = f_0^{\frac{1}{2}} = o(1)$. As we have already discussed, we will regard Φ_0 as depending logarithmically on δ; furthermore, we shall take $\Phi_0 = O(1)$ as $\alpha \to 0$. To match these two expansions, the inner expansion must be cast in the outer variables and expanded to order f_0. Because $\tilde{r} = \hat{r}/\delta$, the large $\tilde{r}$ expansion of φ_0 and φ_1 will be useful. The resultant expansions are

$$\varphi_0 \sim \delta \frac{F}{2\pi}\frac{\sin\omega}{\hat{r}} + O\left(\frac{\delta^2}{\hat{r}^2}\right),$$

$$\varphi_1 \sim \frac{S'}{2\pi}\ln\frac{\hat{r}}{\delta} + \mathscr{K} + \frac{\gamma+1}{2}\frac{(F'^2)'}{16\pi^2}\left(\ln^2\frac{\hat{r}}{\delta} + \cos^2\omega\right) + O\left(\frac{\delta}{\hat{r}}\ln\frac{\hat{r}}{\delta}, \frac{\delta}{\hat{r}}\right).$$

We now introduce

$$\mathscr{K}^* \equiv \mathscr{K} - \frac{S'}{2\pi}\ln\delta + \frac{\gamma+1}{2}\frac{(F'^2)'}{16\pi^2}\ln^2\delta;$$

in order that $\Phi_0 = O(1)$ as $\alpha \to 0$, we require that $\mathscr{K}^* = O(1)$ as $\alpha \to 0$. Thus, the outer expansion of the inner solution reads:

$$(\varphi^i)^0 \sim \alpha\delta\frac{F}{2\pi}\frac{\sin\omega}{\hat{r}} + \alpha^2\left\{\left[\frac{S'}{2\pi} - (\gamma+1)\frac{(F'^2)'}{16\pi^2}\ln\delta\right]\ln\hat{r} + \mathscr{K}^* + \frac{\gamma+1}{2}\frac{(F'^2)'}{16\pi^2}(\ln^2\hat{r} + \cos^2\omega)\right\}. \quad (16)$$

Here we have dropped the terms of order $\alpha\delta^2 = \alpha f_0$ which resulted from the expansion of φ_0 and the terms of order $\alpha^2\delta\ln\delta$ and $\delta\alpha^2$ which resulted from the expansion of φ_1. The first of these is clearly $o(f_0)$ and, if we anticipate the result, $\alpha^2 = \delta^2/|\ln\delta|$ as discussed in § 3, the second set of terms is also seen to be $o(f_0)$.

When the outer solution is written in terms of the inner variables $\tilde{r}$, ω, $\tilde{x}$ we have

$$(\varphi^0)^i = f_0\,\Phi_0(\tilde{x}, \delta\tilde{r}, \omega); \quad (17)$$

thus the boundary condition for the outer problem is applied as $\hat{r} \to 0$. The matching principle requires that (16) and (17) match as $\alpha \to 0$; hence

$$f_0\,\Phi_0(\tilde{x}, \hat{r}, \omega) \sim \delta\alpha\frac{F}{2\pi}\frac{\sin\omega}{\hat{r}} + \alpha^2\left\{\left[\frac{S'}{2\pi} - (\gamma+1)\frac{(F'^2)'}{16\pi^2}\ln\delta\right]\ln\hat{r} + \mathscr{K}^* + \frac{(\gamma+1)}{2}\frac{(F'^2)'}{16\pi^2}(\ln^2\hat{r} + \cos^2\omega)\right\}. \quad (18)$$

Fraenkel (1969) has pointed out that terms containing logarithms, *viz.* the term having α^2 as a coefficient in (18), should be matched as a single term. With this in mind, we see that the appropriate choice for $f_0 = \delta^2$ is $\alpha^2|\ln\delta|$, which further implies that $\delta = \delta(\alpha)$ is given implicitly by

$$\alpha^2 = \delta^2/|\ln\delta|. \quad (19)$$

Thus, the matching requires that, as $\hat{r} \to 0$,

$$\Phi_0(\tilde{x}, \hat{r}, \omega) \sim \frac{1}{|\ln\delta|^{\frac{1}{2}}}\frac{F}{2\pi}\frac{\sin\omega}{\hat{r}} + \left[\frac{S'}{2\pi}\frac{1}{|\ln\delta|} + (\gamma+1)\frac{(F'^2)'}{16\pi^2}\right]\ln\hat{r} + \frac{\mathscr{K}^*}{|\ln\delta|} + \frac{1}{|\ln\delta|}\frac{\gamma+1}{2}\frac{(F'^2)'}{16\pi^2}(\ln^2\hat{r} + \cos^2\omega). \quad (20)$$

According to the analysis presented so far, this is the boundary condition for the outer problem. When the third-, fourth-, and higher-order terms of the inner expansion are calculated, they also make a contribution to this boundary condition; in fact, the actual boundary condition is an infinite sum of terms. This is to be expected as we seek the asymptotic expansion of Φ_0 as $\hat{r} \to 0$ rather than its value at $\hat{r} = 0$. As an example, we could continue the inner expansion (6) to include third- and fourth-order terms

$$\varphi^i = g_0 \varphi_0 + g_1 \varphi_1 + g_2 \varphi_2 + g_3 \varphi_3 + o(g_3)$$

where an inspection of equations (12) and (13) shows that $g_2 = (M_0^2 - 1)\alpha$ and $g_3 = \alpha^3$. The procedure of this paper could be applied to these higher-order terms to determine their contribution to the boundary condition (20). It may be shown that when φ_2 and φ_3 are included in the inner expansion we must add the following quantity to (20):

$$\frac{k}{|\ln\delta|^{\frac{1}{2}}}\left\{\frac{F''}{4\pi}\hat{r}\sin\omega\ln\hat{r} + \delta\mathscr{H}_2'\right\} + \frac{1}{|\ln\delta|^{\frac{3}{2}}}\left\{\frac{\gamma+1}{8\pi}\hat{r}\sin\omega[\tfrac{2}{3}(F'B')'\ln^3\hat{r}\right.$$
$$+ (F'(A' - B' - 2\ln\delta B'))'\ln^2\hat{r} + (F'(\tfrac{3}{2}B' - A' + 2\mathscr{K}^{*\prime}$$
$$\left. + B'\ln\delta))'\ln\hat{r} - (F'B')'\cos^2\omega] + \delta\mathscr{H}_3^*\right\}.$$

Here $k \equiv M_0^2 - 1/\delta^2$, $A \equiv S'/2\pi$, $B \equiv \gamma + 1(F'^2)'/32\pi^2$ and $\mathscr{H}_2^*$ and $\mathscr{H}_3^*$ are harmonic functions proportional to $\hat{r}/\delta$. In like manner we could also determine the contributions of higher-order terms; these contribute terms of even higher order in $\hat{r}$. In order to simplify the boundary condition for the outer problem, we now make the assumption that the outer problem is well-posed provided that the singularities in Φ_0 at $\hat{r} = 0$ are specified. Because the higher-order terms, i.e. the terms $g_i \varphi_i$, $i \geqslant 2$, in the inner expansion contribute terms which vanish as $\hat{r} \to 0$, we may truncate the boundary condition to include only those shown in equation (20). Thus, the outer problem may be written

$$\Phi_{0\hat{r}\hat{r}} + \frac{1}{\hat{r}}\Phi_{0\hat{r}} + \frac{1}{\hat{r}^2}\Phi_{0\hat{r}\hat{r}} = \frac{M_0^2 - 1}{\delta^2}\Phi_{0\tilde{x}\tilde{x}} + (\gamma+1)\Phi_{0\tilde{x}}\Phi_{0\tilde{x}\tilde{x}},$$

where, as $\hat{r} \to 0$,

$$\Phi_0(\tilde{x}, \hat{r}, \omega) \sim \frac{1}{|\ln\delta|^{\frac{1}{2}}}\frac{F}{2\pi}\frac{\sin\omega}{\hat{r}} + \left[\frac{S'}{2\pi}\frac{1}{|\ln\delta|} + (\gamma+1)\frac{(F'^2)'}{16\pi^2}\right]\ln\hat{r}$$
$$+ \frac{\mathscr{K}^*}{|\ln\delta|} + \frac{1}{|\ln\delta|}\left(\frac{\gamma+1}{2}\right)\frac{(F'^2)'}{16\pi^2}(\ln^2\hat{r} + \cos^2\omega), \qquad (21)$$

and, as $\hat{r} \to \infty$, $\hat{r}^{-1}\Phi_{0\omega}$, $\Phi_{0\hat{r}}$, $\Phi_{0\tilde{x}} \to 0$.

Here we recognize the first term as a doublet and the second term as a source having strength $(S'/2\pi)(1/|\ln\delta|) + (\gamma+1)(F'^2)'/16\pi^2$. The first part of the source is due to the nature of the second-order velocity perturbations on the wing and the part depending nonlinearly on the lift is due to the fact that the flow in the neighbourhood of the wing, i.e. $\tilde{r} = O(1)$, appears as a source flow when viewed from the far field. We note also that in theories of transonic flow not involving lift, the solution to the outer problem only depends on $M_0^2 - 1/(\gamma+1)\delta^2$, i.e. the similarity parameter of the problem. Here the solution also depends on $(\gamma+1)$ and $\ln\delta$; hence, in lift dominated flows, no simple similarity rule holds. Furthermore, it is clear that no conventional area or equivalence rule applies for the wings treated here. We refer the reader to Cheng & Hafez (1975) and Cheng (1977) for a further discussion of equivalence rules applied to lifting wings.

Throughout this paper we have confined ourselves to wings having zero thickness. The effect of a wing's thickness is easily incorporated and we will now give a brief discussion of it. The equation of a wing having thickness can be written

$$\tilde{z} = \alpha Z \pm \tau Z_v$$

where the subscript v will always denote functions associated with thickness effects. The inner expansion corresponding to (6) would be

$$\varphi^i = \alpha\varphi_0 + \tau\varphi_v + \alpha^2\varphi_1 + \dots .$$

If we now proceed as we did in § 3, we should find that

$$\varphi_v = \frac{1}{\pi}\int_{a_1}^{a_2} Z_{vx} \ln r_1 \, dy_1 + \mathscr{K}_v,$$

and, as $\tilde{r} \to \infty$,

$$\varphi_v \sim \frac{1}{2\pi} S_v \ln \tilde{r} + \mathscr{K}_v + O\left(\frac{1}{\tilde{r}}\right),$$

where

$$S_v \equiv 2\int_{a_1}^{a_2} Z_{v\tilde{x}} \, dy_1.$$

When this is cast in terms of the outer variables we have

$$(\varphi_v)^0 \sim \frac{1}{2\pi} S_v \ln \hat{r} + \mathscr{K}_v^* + O\left(\frac{\delta}{\hat{r}}\right),$$

where $\mathscr{K}_v^* \equiv \mathscr{K}_v - (S_v/2\pi)\ln\delta \equiv O(1)$. Thus, the thickness would contribute

$$\tau[(S_v/2\pi)\ln\hat{r} + \mathscr{K}_v^*]$$

to equation (18). Inspection shows that the thickness and lift have an equal effect on the outer problem provided $\tau = O(f_0) = O(\delta^2)$, where δ is related to α by (19). An examination of higher-order terms shows that this is the only additional singularity generated by the introduction of the thickness; thus, provided $\tau = O(\delta^2)$, the thickness contributes $\tau/\delta^2[S_v/2\pi \ln\hat{r} + \mathscr{K}_v^*]$ to the boundary condition in (21). Generally, when $\tau \neq O(\delta^2(\alpha))$, where $\delta(\alpha)$ is given by (19), we may neglect either the lift or thickness when calculating the outer flow. For example, when $\tau = O(\alpha)$, the matching requires that $f_0 = \tau$, $\delta = \tau^{\frac{1}{2}}$ and that the boundary condition is

$$\Phi_0 \sim \frac{S_v}{2\pi}\ln\hat{r} + \mathscr{K}_v^*$$

as $\hat{r} \to 0$. When $\tau = O(\alpha^3)$, the matching yields the same results as in the zero thickness case; in this case the thickness effects may be considered negligible for the purposes of calculating the far field.

6. Discussion of previous investigations

In this section we discuss the investigations of Barnwell (1975) and Cheng & Hafez (1975), comparing their results and procedures to ours. Both papers give derivations of the boundary-value problem governing the flow far from a lifting wing; their procedures are seen to differ considerably in both appearance and content from each other and the present study.

We first discuss the work of Barnwell. It should first be mentioned that Barnwell uses a body oriented co-ordinate system in contrast to the wind oriented system used here; hence care should be exercised in comparing Barnwell's work to either ours or Cheng & Hafez's. Barnwell also provides a discussion of the effect of leading edge separation; this complication will not be discussed here. Barnwell first presents an inner expansion which contains gauge functions which are logarithmic in the ratio of the inner and outer length scales. Although our inner expansion proceeded in integral powers of α and M_0^2-1, we allowed $\mathscr{K}$ and $\mathscr{K}_H(\tilde{y},\tilde{z})$ in (9) to depend on α; thus, the resultant inner expansion is seen to be equivalent to that of Barnwell. A further examination of Barnwell's inner expansion shows that Barnwell has omitted the following term,

$$\frac{\tilde{r}}{\lambda^4}\frac{\partial^2\phi_1}{\partial\tilde{z}^2}g = -\frac{\tilde{r}}{\lambda^4}\frac{\partial^2\phi_1}{\partial\tilde{y}^2}g,$$

from his boundary condition (19). This produces an error in the strength of the equivalent source given by $H(\bar{x})$ in his equation (68).

Barnwell also presents a very careful study of the outer solution. As we mentioned in § 4 he finds the equations governing three terms of the outer expansion; in his notation these terms are

$$\epsilon_1\,\Phi_1+\epsilon_2\,\Phi_2+\epsilon_3\,\Phi_3,$$

where ϵ_1, ϵ_2 and ϵ_3 are the outer gauge functions and Φ_1, Φ_2 and Φ_3 are independent of any small parameters. The lowest-order term satisfies the small disturbance transonic equation and Φ_2 and Φ_3 satisfy linear equations which have coefficients dependent on the lower-order Φ_i's and their derivatives. To match the inner and outer expansions he needs a small $\bar{r}$ expansion of the Φ_i's. To obtain this he uses the iterative technique of Cole & Messiter (1957) to solve the differential equations governing Φ_1, Φ_2 and Φ_3 for small values of $\bar{r}$; this assures us that the inner expansion of the outer solution satisfies the outer equations. For the sake of simplicity, we have presented a more intuitive approach to this than that presented by Barnwell. Essentially, we have tacitly assumed that a small $\hat{r}$ expansion of our outer solution will contain all the terms necessary to match. It is easy to show that when such an iterative procedure is applied to our outer solution, a boundary condition results which is identical to the one presented here. Once Barnwell obtains his expansion of the outer solution he matches this to the large $\tilde{r}$ expansion of the inner solution. Except for the error in the source strength mentioned above his results are in agreement with those given here.

As a final remark we note that Barnwell states that an intermediate expansion is necessary in order to match the inner and outer expansions. He bases this on an examination of the large $\tilde{r}$ expansion of the inner solution (his equation 67) and the small $\bar{r}$ expansion of the outer solution (his equation 68). Because the leading term in (67) is a dipole and the leading term in (68) is a source, he concludes that an intermediate expansion is necessary. In § 5 we used a rule concerning the matching of logarithms; if this is applied to Barnwell's expansions (67) and (68) it is clear that they may be matched without recourse to an intermediate expansion.

We now discuss the results of Cheng & Hafez. Of the two previous investigations the procedure of Cheng & Hafez has the closest resemblance to ours. Their inner expansion can be shown to be the same as ours and they use a one-term outer expansion similar to that given here. Throughout their paper, Cheng & Hafez use an elaborate parameterization scheme. They also correctly state that their results are valid for $\lambda \equiv b/l = O(1)$; this appears to be inconsistent with the parameterization scheme. Specifically, in equation (2.10) they introduce a parameter

$$\Gamma_* \equiv 8/(\gamma+1)\,\lambda^2|\ln \epsilon|,$$

where ϵ is the ratio of inner and outer length scales analogous to our δ; they further require that Γ_* be non-vanishing as $\epsilon \to 0$. This would seem to imply that λ must vary as $|\ln \epsilon|^{-\frac{1}{2}}$ which violates their $\lambda = O(1)$ assumption. However, this inconsistency in the parameterization does not affect the final results.

In § 4.3 the outer equation is introduced and a small $\eta \equiv \epsilon r$ expansion of the outer solution is given. In § 4.5 the matching is carried out for the case corresponding to the one discussed here. The boundary condition for the outer problem is given by their equation (4.12); this is seen to disagree with our boundary condition (21). Specifically, the terms

$$\frac{\epsilon'}{2\pi}\left[\bar{D}_1(x)\,(\eta')^{-1}\cos\omega + |\ln \epsilon'|^{-\frac{1}{2}} m_{32}(\eta')^{-2}\sin 2\omega\right]$$

appear in their equation (4.12), but are absent in ours. It may be shown that these terms correspond to the $O(\delta^2/\hat{r}^2)$ term found in the outer expansion of φ_0 and the $O(\delta\hat{r}^{-1}\ln\hat{r}/\delta, \delta/\hat{r})$ term found in the outer expansion of φ_1; these higher-order terms must be truncated in the matching. In a later publication, Cheng (1977) discusses the application of this theory to particular wing configurations; the boundary condition used in this study is equivalent to the one derived here.

With the exception of the errors mentioned above, the results of Barnwell (1975), Cheng & Hafez (1975) and the present study are in agreement. The study presented here approaches the problem from a more fundamental point of view and is therefore believed to be more accessible to the reader.

7. Conclusion

We have presented a theory of thin three-dimensional wings without thickness in transonic flow. The boundary-value problem governing the flow far from the wing has been derived. The calculations presented here are intended to be simpler than those of the previously published studies; they also correct errors found in these earlier studies. Both the previous investigations and the present study show that there are effects which are of second order in the near-field which produce first-order effects in the far-field.

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Appendix A. Solution for φ_1

Because of the linearity of the Laplacian it is permissible to break φ_1 up into three parts:

$$\varphi_1 = \varphi' + \varphi'' + \tfrac{1}{2}(\gamma+1)\,\varphi''',$$

where φ' satisfies

$$\nabla_2^2\,\varphi' = 0$$

with $\varphi'_{\tilde{z}} = (\varphi_{0\tilde{y}}\,Z)_{\tilde{y}}$ on $\tilde{z} = 0\pm$, $a_1 \leqslant \tilde{y} \leqslant a_2$;

φ'' satisfies

$$\nabla_2^2\,\varphi'' = \frac{\partial}{\partial \tilde{x}}(\varphi_{0\tilde{y}}^2 + \varphi_{0\tilde{z}}^2)$$

with $\varphi''_{\tilde{z}} = \varphi_{0\tilde{x}}\,Z_{\tilde{x}}$ on $\tilde{z} = 0\pm$, $a_1 \leqslant \tilde{y} \leqslant a_2$,

and φ''' satisfies

$$\nabla_2^2\,\varphi''' = \frac{\partial}{\partial \tilde{x}}(\varphi_{0\tilde{x}}^2)$$

with $\varphi'''_{\tilde{z}} = 0$ on $\tilde{z} = 0\pm$, $a_1 \leqslant \tilde{y} \leqslant a_2$.

Equation (14) admits solutions for φ_1 which are symmetric in $\tilde{z}$, i.e. $\varphi_1(\tilde{z}) = \varphi_1(-\tilde{z})$. In the following we will assume that φ_1 as well as φ', φ'', and φ''' are symmetric in $\tilde{z}$.

The above problem for φ' is homogeneous; thus, we will not only take $[\psi] = 0$, $[\mathscr{H}_H] = 0$ and $f = (\varphi_{0\tilde{y}}\,Z)_{\tilde{y}}$ in (9), but $\Psi = 0$ as well. Thus,

$$\varphi' = \frac{1}{2\pi}\int_{a_1}^{a_2} ([\varphi_{0\tilde{y}}]\,Z)_{y_1} \ln r_1\, dy_1 + \mathscr{K}_1(\tilde{x}) + \mathscr{H}',$$

where $\mathscr{H}'$ is the arbitrary harmonic function found in equation (9). An integration by parts yields

$$\varphi' = \frac{1}{2\pi}\{\ln r_1[\varphi_{0\tilde{y}}]\,Z\}\bigg|_{a_1}^{a_2} - \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_{0\tilde{y}}]\,Z\,\frac{y_1-\tilde{y}}{r_1^2}\,dy_1 + \mathscr{K}_1 + \mathscr{H}'.$$

Here we follow Cheng & Hafez (1975) and require that $[\varphi_{0\tilde{y}}] \equiv 0$ at the leading edges of the wing and the outer edges of the trailing vortex sheet. Thus,

$$\varphi' = \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_{0\tilde{y}}]\,Z\,\frac{\tilde{y}-y_1}{r_1^2}\,dy_1 + \mathscr{K}_1(\tilde{x}) + \mathscr{H}'. \tag{A 1}$$

The large $\tilde{r}$ behaviour of φ' is

$$\varphi' \sim \mathscr{K}_1(\tilde{x}) + \frac{\cos\omega}{2\pi\tilde{r}}\int_{a_1}^{a_2} [\varphi_{0\tilde{y}}]\,Z\,dy_1 + \mathscr{H}'(\tilde{r},\omega;\tilde{x}) + O\left(\frac{1}{\tilde{r}^2}\right). \tag{A 2}$$

In the large $\tilde{r}$ expansions of φ', φ'' and φ''', we will not expand the arbitrary harmonic functions; this behaviour must be obtained from the matching.

We now derive the solution for φ''. The function $\mathscr{F}(\zeta,\bar{\zeta};\tilde{x})$ in § 3 is seen to be equal to $4(\varphi_{0\zeta}\,\varphi_{0\bar{\zeta}})_{\tilde{x}}$; Cheng & Hafez (1973) have shown that when the arbitrary function $\mathscr{H}_p$ in § 3 is taken to be identically zero, the function Ψ, or here Ψ'', is given by

$$\Psi'' = (\tfrac{1}{2}\varphi_0^2)_{\tilde{x}},$$

and, from the boundary condition for φ_0, we have

$$\Psi''_{\tilde z}(\tilde y, 0\pm;\tilde x) = \varphi_{0\tilde x} Z_{\tilde x} + \varphi_0 Z_{\tilde x\tilde x}.$$

Thus, the solution for φ'' which is symmetric in $\tilde z$ is

$$\varphi'' = (\tfrac{1}{2}\varphi_0^2)_{\tilde x} - \frac{1}{2\pi}\int_{a_1}^{a_2} [\varphi_0] Z_{\tilde x\tilde x} \ln r_1 \, dy_1 + \mathscr{K}_2(\tilde x) + \mathscr{H}''. \tag{A 3}$$

For large $\tilde r$,

$$\varphi'' \sim \frac{1}{2\pi} G(\tilde x) \ln \tilde r + \mathscr{K}_2(\tilde x) + \mathscr{H}'' + O\left(\frac{1}{\tilde r}\right), \tag{A 4}$$

where

$$G(\tilde x) \equiv -\int_{a_1}^{a_2} [\varphi_0] Z_{\tilde x\tilde x} \, dy_1.$$

Finally, we consider the problem for φ'''. The function $\mathscr{F}(\zeta, \bar\zeta; \tilde x)$ in § 3 is seen to be $\partial(\varphi_{0\tilde x}^2(\zeta, \bar\zeta; \tilde x))/\partial \tilde x$, where

$$\varphi_{0\tilde x}^2 = -\frac{1}{16\pi^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} [\varphi_{0\tilde x}]_1 [\varphi_{0\tilde x}]_2 \left\{\frac{1}{y_1-\zeta} - \frac{1}{y_1-\bar\zeta}\right\}\left\{\frac{1}{y_2-\zeta} - \frac{1}{y_2-\bar\zeta}\right\} dy_1\, dy_2,$$

where $[\varphi_{0\tilde x}]_i \equiv [\varphi_{0\tilde x}]\,(y_i; \tilde x)$, $i = 1, 2$. Cheng & Hafez (1973) have shown that Ψ, or here Ψ''', can be written

$$\Psi''' = -\frac{1}{64\pi^2}\frac{\partial}{\partial \tilde x}\int_{a_1}^{a_2}\int_{a_1}^{a_2} \frac{[\varphi_{0\tilde x}]_1 [\varphi_{0\tilde x}]_2}{y_1 - y_2} \ln\left(\frac{\zeta - y_1}{\zeta - y_2}\cdot\frac{\bar\zeta - y_1}{\bar\zeta - y_2}\right) dy_1\, dy_2 + \frac{1}{64\pi^2}\frac{\partial}{\partial \tilde x}\left\{\int_{a_1}^{a_2} [\varphi_{0\tilde x}]_1 \ln(\zeta - y_1)(\bar\zeta - y_1)\, dy_1\right\}^2,$$

provided that we choose $\mathscr{H}_p$ (see § 3) as follows

$$\mathscr{H}_p = \frac{1}{64\pi^2}\frac{\partial}{\partial \tilde x}\int_{a_1}^{a_2}\int_{a_1}^{a_2} [\varphi_{0\tilde x}]_1 [\varphi_{0\tilde x}]_2 \{\ln(\zeta - y_1)(\zeta - y_2) + \ln(\bar\zeta - y_1)(\bar\zeta - y_2)\}\, dy_1\, dy_2 - \frac{1}{64\pi^2}\frac{\partial}{\partial \tilde x}\int_{a_1}^{a_2}\int_{a_1}^{a_2} \frac{[\varphi_{0\tilde x}]_1 [\varphi_{0\tilde x}]_2}{y_1 - y_2}\left\{\zeta \ln\left(\frac{\zeta - y_1}{\zeta - y_2}\right) + \bar\zeta \ln\left(\frac{\bar\zeta - y_1}{\bar\zeta - y_2}\right)\right\} dy_1\, dy_2.$$

At $\tilde z = 0\pm$, Cheng & Hafez (1973) have also shown that

$$\Psi'''_{\tilde z}(\tilde y, 0\pm, \tilde x) = \pm\frac{1}{2\pi}\frac{\partial}{\partial \tilde x}\left\{[\varphi_{0\tilde x}](\tilde y; \tilde x)\, \text{P.V.}\int_{a_2}^{a_2} [\varphi_{0\tilde x}]_1 \left(\ln|y_1 - \tilde y| + \frac{\tilde y}{y_1 - \tilde y}\right) dy_1\right\}, \tag{A 5}$$

where the P.V. indicates that the Cauchy Principal Value of the integral is to be taken. Thus, the solution for φ''' which is symmetric in $\tilde z$ is

$$\varphi''' = \Psi''' - \frac{1}{2\pi}\int_{a_1}^{a_2} [\Psi'''_z] \ln r_1\, dy_1 + \mathscr{K}_3(\tilde x) + \mathscr{H}''', \tag{A 6}$$

where, in terms of the real variables $\tilde y$ and $\tilde z$, Ψ''' is

$$\Psi''' = -\frac{\tilde y}{16\pi^2}\frac{\partial}{\partial \tilde x}\left\{\int_{a_1}^{a_2}\int_{a_1}^{a_2} \frac{[\varphi_{0\tilde x}]_1 [\varphi_{0\tilde x}]_2}{y_1 - y_2} \ln\frac{r_1}{r_2}\, dy_1\, dy_2\right\} + \frac{1}{16\pi^2}\frac{\partial}{\partial \tilde x}\left\{\int_{a_1}^{a_2} [\varphi_{0\tilde x}]_1 \ln r_1\, dy_1\right\}^2, \tag{A 7}$$

and $\Psi'''_{\tilde z}(\tilde y, 0\pm; \tilde x)$ is given by (A 5). For large values of $\tilde r$, Ψ''' has the behaviour

$$\Psi''' \sim \frac{1}{16\pi^2}(\ln^2 \tilde r + \cos^2 \omega)(F'^2)' + O\left(\frac{\ln \tilde r}{\tilde r}, \frac{1}{\tilde r}\right),$$

where $F' \equiv (d/d\tilde{x})\,F(\tilde{x})$. Thus, as $\tilde{r}\to\infty$

$$\varphi''' \sim \frac{1}{16\pi^2}(\ln^2\tilde{r}+\cos^2\omega)\,(F'^2)'+\frac{1}{2\pi}I(\tilde{x})\ln\tilde{r}+\mathscr{K}_3+\mathscr{H}'''+O\left(\frac{\ln\tilde{r}}{\tilde{r}},\frac{1}{\tilde{r}}\right), \quad \text{(A 8)}$$

where

$$I(\tilde{x}) \equiv -\int_{a_1}^{a_2}[\Psi_{\tilde{z}}''''']\,dy_1.$$

Thus, φ_1 is given by the sum of the terms φ', φ'' and $\frac{1}{2}(\gamma+1)\,\varphi'''$. The large $\tilde{r}$ behaviour of φ_1 may be obtained by equations (A 2), (A 4) and (A 8); this may be written

$$\varphi_1 \sim \frac{S'(\tilde{x})}{2\pi}\ln\tilde{r}+\mathscr{K}(\tilde{x})+\frac{\gamma+1}{32\pi^2}(F'^2)'(\ln^2\tilde{r}+\cos^2\omega)+\mathscr{H}_1+O\left(\frac{\ln\tilde{r}}{\tilde{r}},\frac{1}{\tilde{r}}\right),$$

where

$$S'(\tilde{x}) \equiv \frac{dS(\tilde{x})}{d\tilde{x}} \equiv G(\tilde{x})+\frac{\gamma+1}{2}I(\tilde{x}),$$

$$\mathscr{K}(\tilde{x}) \equiv \mathscr{K}_1+\mathscr{K}_2+\frac{\gamma+1}{2}\mathscr{K}_3$$

and

$$\mathscr{H}_1(\tilde{r},\omega;\tilde{x}) \equiv \mathscr{H}'+\mathscr{H}''+\frac{\gamma+1}{2}\mathscr{H}'''.$$

As we did in the discussion of φ_0, we will now anticipate some of the results of the matching to determine $\mathscr{H}_1$ for all values of $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. We will require that the boundary condition for the outer problem contains contributions from φ', φ'' and φ'''; the only way that this will be possible is $\mathscr{H}_1 = O(\ln\tilde{r})$ at most, as $\tilde{r}\to\infty$. This condition, combined with the fact that $\mathscr{H}_1$ is symmetric in $\tilde{z}$ and satisfies

$$\nabla_2^2\mathscr{H}_1 = 0 \quad \text{for all} \quad \tilde{x},\tilde{y},\tilde{z},$$

and $$\mathscr{H}_{1\tilde{z}} = 0 \quad \text{on} \quad \tilde{z}=0\pm, \quad a_1 \leqslant \tilde{y} \leqslant a_2,$$

implies that $\mathscr{H}_1$ is a function of $\tilde{x}$ alone. If we absorb this function of $\tilde{x}$ in $\mathscr{K}(\tilde{x})$ we may now write

$$\varphi_1 = \frac{1}{2\pi}\int_{a_1}^{a_2}[\varphi_{0\tilde{y}}]\,Z\frac{\tilde{y}-y_1}{r_1^2}\,dy_1+\tfrac{1}{2}(\varphi_0^2)_{\tilde{x}}-\frac{1}{2\pi}\int_{a_1}^{a_2}[\varphi_0]\,Z_{\tilde{x}\tilde{x}}\ln r_1\,dy_1$$
$$+\frac{\gamma+1}{2}\left\{\Psi'''-\frac{1}{2\pi}\int_{a_1}^{a_2}[\Psi_{\tilde{z}}''''']\ln r_1\,dy_1\right\}+\mathscr{K}(\tilde{x}), \quad \text{(A 9)}$$

and as $\tilde{r}\to\infty$,

$$\varphi_1 \sim \frac{S'(\tilde{x})}{2\pi}\ln\tilde{r}+\mathscr{K}(\tilde{x})+\frac{\gamma+1}{32\pi^2}(F'^2)'(\ln^2\tilde{r}+\cos^2\omega)+O\left(\frac{\ln\tilde{r}}{\tilde{r}},\frac{1}{\tilde{r}}\right). \quad \text{(A 10)}$$

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